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IMAGE FORCE ON LINE DISLOCATIONS IN ANISOTROPIC ELASTIC HALF-SPACES WITH A FIXED BOUNDARY

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Abstract—Employing explicit expressions of Green's functions for an anisotropic elastic half-space, the image force exerted on a line dislocation in the half-space is obtained. When the boundary of the half-space is traction free, the image force obtained agrees with that existing in the literature. We also obtain the image force for the case in which the boundary is fixed (i.e. clamped). In the special case of isotropic materials the result reduces to that available in the literature. While the image force for the traction-free boundary always tends to attract the dislocation to the boundary, the image force for the fixed boundary is repulsive. Moreover, the magnitude of this repulsive force is always equal to or larger than that of the attractive force associated with a free boundary. We also show how the image force for the half-space can be deduced from the image force for two joined dissimilar half-spaces. The image force is independent of the orientation of the half-space boundary once the Burgers vector of the dislocation is prescribed, provided that the boundary is parallel to the x_3-axis and the distance between the dislocation and the boundary is not altered. This agrees with the more general case of the image force theorem for two joined dissimilar half-spaces, one of which is dislocated.

1. INTRODUCTION

The image force, or the Peach–Koehler force, exerted on a line dislocation in an anisotropic elastic half-space in which the boundary of the half-space is traction-free has been previously obtained, as has the image force on a line dislocation in anisotropic bimaterials (Barnett and Lothe, 1974; Rice, 1985). The corresponding problem for the half-space when the boundary is a fixed (i.e. clamped) surface does not seem to have been as thoroughly studied, although the result for the associated problem of a dislocated isotropic elastic half-space is known (Dundurs, 1969; Mura, 1987). It should be mentioned that Tucker (1969) has studied a variety of problems involving dislocations interacting with straight boundaries in anisotropic half-spaces and bimaterials. Tucker's method is based on Stroh's 1958 paper and thus does not make use of the normalization inherent in eqn (12) introduced by Stroh in 1962. As a result, Tucker's solutions are somewhat less transparent than those presented here. We present an explicit expression for the image force when the boundary is fixed and show that the magnitude of the image force is equal to or larger than the image force for the case when the boundary is traction-free.

After presenting briefly the Stroh (1958, 1962) formalism for two-dimensional anisotropic elasticity and certain identities needed in Section 2, the Green's functions for the half-space $x_2 > 0$ with the boundary $x_2 = 0$ being traction-free or fixed are given in Section 3. The Green's functions are explicit; there are no integrals required to calculate stress and/or displacement at any point in the half-space. Using these explicit Green's functions, the image force which a fixed boundary exerts on a line dislocation in the half-space is derived in Section 4. While the image force is always attractive if the boundary is traction-free, it is repulsive when the boundary is a fixed boundary, and, the magnitude of the image force for the fixed boundary is equal to or larger than that for the traction-free boundary. Whether the boundary is fixed or traction-free, the image force is invariant in the sense that the image force is the same for all boundaries parallel to the dislocation line if the distance between the line dislocation and the boundary is unaltered. In Section 5 we briefly rederive the image force for dislocated bimaterials and show that the image force for a dislocated half-space with either fixed or free boundary can be deduced by taking a proper limit as the elastic stiffnesses of the undislocated half-space in the bimaterial tend to either infinity or zero, respectively.

2. THE BASIC EQUATIONS

In a fixed rectangular coordinate system x_i ($i = 1, 2, 3$) let u_i, σ_{ij} be, respectively, the displacement and stress in an anisotropic elastic material. The stress–strain laws and the equations of equilibrium are

$$\sigma_{ij} = C_{ijks}u_{k,s}, \tag{1}$$

$$C_{ijks}u_{k,sj} = 0, \tag{2}$$

where a comma stands for differentiation, repeated indices imply summation and the C_{ijks} are the elasticity constants which are assumed to be fully symmetric and positive definite. For two-dimensional deformations in which u_i depends on x_1, x_2 only, a general solution to (2) is, in matrix notation,

$$\mathbf{u} = \mathbf{a}f(z), \quad z = x_1 + px_2. \tag{3}$$

In the above, f is an arbitrary function of z, and p and $\mathbf{a}$ are determined by inserting (3) into (2). We have

$$\{\mathbf{Q} + p(\mathbf{R} + \mathbf{R}^{\mathrm{T}}) + p^2\mathbf{T}\}\mathbf{a} = 0, \tag{4}$$

where the superscript T denotes the transpose and $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{T}$ are 3×3 real matrices whose components are

$$Q_{ik} = C_{i1k1}, \quad R_{ik} = C_{i1k2}, \quad T_{ik} = C_{i2k2}.$$

The stresses obtained by substituting $(3)_1$ into (1) can be written in terms of the stress function $\boldsymbol{\phi}$ as

$$\sigma_{1i} = -\phi_{i,2}, \quad \sigma_{2i} = \phi_{i,1}, \tag{5}$$

in which

$$\boldsymbol{\phi} = \mathbf{b}f(z), \tag{6}$$

$$\mathbf{b} = (\mathbf{R}^{\mathrm{T}} + p\mathbf{T})\mathbf{a} = -\frac{1}{p}(\mathbf{Q} + p\mathbf{R})\mathbf{a}. \tag{7}$$

The second equality in (7) follows from (4). It suffices therefore to consider the stress function $\boldsymbol{\phi}$ because the stresses σ_{ij} can be obtained by differentiation.

There are six eigenvalues p from (4) which consist of three pairs of complex conjugates (Eshelby *et al.*, 1953; Stroh, 1958). If p_α, $\mathbf{a}_\alpha$ ($\alpha = 1, 2, \ldots, 6$) are the eigenvalues and the associated eigenvectors, we let

$$\operatorname{Im} p_\alpha > 0, \quad p_{\alpha+3} = \bar{p}_\alpha, \quad \mathbf{a}_{\alpha+3} = \bar{\mathbf{a}}_\alpha, \quad \mathbf{b}_{\alpha+3} = \bar{\mathbf{b}}_\alpha, \quad (\alpha = 1, 2, 3),$$

where Im stands for the imaginary part and the overbar denotes the complex conjugate.

Assuming that the p_α are distinct, the general solution for $\mathbf{u}$ and $\boldsymbol{\phi}$ obtained by superposing six solutions of the form (3) and (6) are

$$\mathbf{u} = \sum_{\alpha=1}^{3} \{\mathbf{a}_\alpha f_\alpha(z_\alpha) + \bar{\mathbf{a}}_\alpha f_{\alpha+3}(\bar{z}_\alpha)\},$$
$$\boldsymbol{\phi} = \sum_{\alpha=1}^{3} \{\mathbf{b}_\alpha f_\alpha(z_\alpha) + \bar{\mathbf{b}}_\alpha f_{\alpha+3}(\bar{z}_\alpha)\}. \tag{8}$$

In (8) $f_1, f_2, \ldots, f_6$ are arbitrary functions of their argument and

$$z_\alpha = x_1 + p_\alpha x_2.$$

In most applications the f_α assume the same function form so that we may write

$$f_\alpha(z_\alpha) = q_\alpha f(z_\alpha), \quad f_{\alpha+3}(\bar{z}_\alpha) = \bar{q}_\alpha \bar{f}(\bar{z}_\alpha), \quad \alpha = 1,2,3,$$

where the q_α are complex constants. The second equation is useful for obtaining real solutions for $\mathbf{u}$ and $\boldsymbol{\phi}$. Equation (8) can then be written as, after replacing q_α by $q_\alpha/2$,

$$\mathbf{u} = \mathrm{Re}\,\{\mathbf{A}\langle f(z_*)\rangle\mathbf{q}\},$$
$$\boldsymbol{\phi} = \mathrm{Re}\,\{\mathbf{B}\langle f(z_*)\rangle\mathbf{q}\}. \tag{9}$$

Here Re stands for the real part, $\mathbf{A}$, $\mathbf{B}$ are the 3×3 complex matrices defined by

$$\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3], \quad \mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3], \tag{10}$$

and $\langle f(z_*)\rangle$ is the diagonal matrix

$$\langle f(z_*)\rangle = \mathrm{diag}\,\{f(z_1), f(z_2), f(z_3)\}.$$

When $\mathbf{q}$ is replaced by $-\mathrm{i}\mathbf{q}$, (9) leads to the alternative form:

$$\mathbf{u} = \mathrm{Im}\,\{\mathbf{A}\langle f(z_*)\rangle\mathbf{q}\},$$
$$\boldsymbol{\phi} = \mathrm{Im}\,\{\mathbf{B}\langle f(z_*)\rangle\mathbf{q}\}. \tag{11}$$

For a given problem all one has to do is to determine the unknown function $f(z_\alpha)$ and the complex constant vector $\mathbf{q}$.

The complex matrices $\mathbf{A}$ and $\mathbf{B}$ of (10), when properly normalized, satisfy the orthogonality relation (Stroh, 1962; Barnett and Lothe, 1973; Chadwick and Smith, 1977):

$$\begin{bmatrix} \mathbf{B}^{\mathrm{T}} & \mathbf{A}^{\mathrm{T}} \\ \bar{\mathbf{B}}^{\mathrm{T}} & \bar{\mathbf{A}}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \bar{\mathbf{A}} \\ \mathbf{B} & \bar{\mathbf{B}} \end{bmatrix} = \mathbf{I}, \tag{12}$$

in which $\mathbf{I}$ is the 6×6 unit matrix. The three matrices $\mathbf{S}$, $\mathbf{H}$, $\mathbf{L}$, defined by

$$\mathbf{S} = \mathrm{i}(2\mathbf{A}\mathbf{B}^{\mathrm{T}} - \mathbf{I}), \quad \mathbf{H} = 2\mathrm{i}\mathbf{A}\mathbf{A}^{\mathrm{T}}, \quad \mathbf{L} = -2\mathrm{i}\mathbf{B}\mathbf{B}^{\mathrm{T}}, \tag{13}$$

can be shown to be real. The matrices $\mathbf{H}$ and $\mathbf{L}$ are symmetric. It can be shown that they are positive definite (Gundersen *et al.*, 1987; Ting, 1988) and that $\mathbf{SH}$, $\mathbf{LS}$, $\mathbf{H}^{-1}\mathbf{S}$ and $\mathbf{SL}^{-1}$

are antisymmetric. Moreover, **S**, **H** and **L** are related by

$$\mathbf{HL}-\mathbf{SS}=\mathbf{I}. \tag{14}$$

A useful quantity which will appear in the analysis is the surface impedance tensor **M** defined by

$$\mathbf{M}=-\mathrm{i}\mathbf{BA}^{-1}=\mathbf{H}^{-1}+\mathrm{i}\mathbf{H}^{-1}\mathbf{S}, \tag{15a}$$

$$\mathbf{M}^{-1}=\mathrm{i}\mathbf{AB}^{-1}=\mathbf{L}^{-1}-\mathrm{i}\mathbf{SL}^{-1}. \tag{15b}$$

The second equalities in (15a, b) are obtained from (13) and the relations

$$\mathbf{BA}^{-1}=(\mathbf{AB}^{\mathrm{T}})^{\mathrm{T}}(\mathbf{AA}^{\mathrm{T}})^{-1},\quad \mathbf{AB}^{-1}=(\mathbf{AB}^{\mathrm{T}})(\mathbf{BB}^{\mathrm{T}})^{-1}.$$

The tensor **M**, and hence $\bar{\mathbf{M}}$, $\mathbf{M}^{-1}$, $\bar{\mathbf{M}}^{-1}$, is Hermitian and it can be proved that these are positive definite (Ingebrigtsen and Tonning, 1969; Lothe and Barnett, 1976; Chadwick and Ting, 1987).

3. THE GREEN'S FUNCTIONS FOR INFINITE SPACES AND HALF-SPACES

For the infinite space subjected to a line force $\boldsymbol{f}$ and a line dislocation with Burgers vector $\boldsymbol{b}$ at the origin $x_1=x_2=0$, the solution in the form of (11) is

$$\mathbf{u}=\frac{1}{\pi}\mathrm{Im}\,\{\mathbf{A}\langle\ln z_*\rangle\mathbf{q}_0\},$$

$$\boldsymbol{\phi}=\frac{1}{\pi}\mathrm{Im}\,\{\mathbf{B}\langle\ln z_*\rangle\mathbf{q}_0\}, \tag{16}$$

where $\mathbf{q}_0$ is a complex constant vector to be determined. Since $(\ln z_\alpha)$ is a multi-valued function we introduce a cut along the negative x_1-axis. In the polar-coordinate system, the solution (16) applies to

$$-\pi<\theta<\pi,\quad r>0.$$

Therefore

$$\ln z_\alpha=\ln r\pm\mathrm{i}\pi\quad \text{at}\ \theta=\pm\pi,\quad \text{for}\ \alpha=1,2,3.$$

Equations (16) must satisfy the conditions

$$\mathbf{u}(\pi)-\mathbf{u}(-\pi)=\boldsymbol{b},\quad \boldsymbol{\phi}(\pi)-\boldsymbol{\phi}(-\pi)=\boldsymbol{f},$$

which lead to

$$2\,\mathrm{Re}\,(\mathbf{Aq}_0)=\boldsymbol{b},\quad 2\,\mathrm{Re}\,(\mathbf{Bq}_0)=\boldsymbol{f}. \tag{17}$$

This can be written as

$$\begin{bmatrix}\mathbf{A} & \bar{\mathbf{A}}\\ \mathbf{B} & \bar{\mathbf{B}}\end{bmatrix}\begin{bmatrix}\mathbf{q}_0\\ \bar{\mathbf{q}}_0\end{bmatrix}=\begin{bmatrix}\boldsymbol{b}\\ \boldsymbol{f}\end{bmatrix}.$$

It follows from (12) that

$$\begin{bmatrix} \mathbf{q}_0 \\ \bar{\mathbf{q}}_0 \end{bmatrix} = \begin{bmatrix} \mathbf{B}^{\mathrm{T}} & \mathbf{A}^{\mathrm{T}} \\ \bar{\mathbf{B}}^{\mathrm{T}} & \bar{\mathbf{A}}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{f} \end{bmatrix}.$$

Hence

$$\mathbf{q}_0 = \mathbf{A}^{\mathrm{T}}\boldsymbol{f} + \mathbf{B}^{\mathrm{T}}\boldsymbol{b}. \tag{18}$$

Consider now the half-space $x_2 > 0$ which is occupied by the material, and a line force $\boldsymbol{f}$ and a line dislocation with Burgers vector $\boldsymbol{b}$ co-existing at

$$(x_1, x_2) = (0, d), \quad d > 0. \tag{19}$$

The solution can be written as (Ting, 1992b)

$$\begin{aligned} \mathbf{u} &= \frac{1}{\pi}\,\mathrm{Im}\,[\mathbf{A}\langle \ln(z_* - p_* d)\rangle \mathbf{q}_0] + \frac{1}{\pi}\,\mathrm{Im}\sum_{\beta=1}^{3}[\mathbf{A}\langle \ln(z_* - \bar{p}_\beta d)\rangle \mathbf{q}_\beta], \\ \boldsymbol{\phi} &= \frac{1}{\pi}\,\mathrm{Im}\,[\mathbf{B}\langle \ln(z_* - p_* d)\rangle \mathbf{q}_0] + \frac{1}{\pi}\,\mathrm{Im}\sum_{\beta=1}^{3}[\mathbf{B}\langle \ln(z_* - \bar{p}_\beta d)\rangle \mathbf{q}_\beta], \end{aligned} \tag{20}$$

where $\mathbf{q}_0$ is given in (18) and $\mathbf{q}_\beta$ are unknown constant vectors to be determined. In the above,

$$\begin{aligned} \langle \ln(z_* - p_* d)\rangle &= \mathrm{diag}\,[\ln(z_1 - p_1 d), \ln(z_2 - p_2 d), \ln(z_3 - p_3 d)], \\ \langle \ln(z_* - \bar{p}_\beta d)\rangle &= \mathrm{diag}\,[\ln(z_1 - \bar{p}_\beta d), \ln(z_2 - \bar{p}_\beta d), \ln(z_3 - \bar{p}_\beta d)]. \end{aligned}$$

Thus the asterisk (*) is for identifying the elements in the diagonal matrix. The first terms on the right of (20) represent the Green's function for an infinite space with the singularities at the location given by (19). The second terms in (20) are added to satisfy the boundary condition at $x_2 = 0$.

Consider first the case in which the surface $x_2 = 0$ is traction-free so that

$$\boldsymbol{\phi} = 0, \quad \text{at}\, x_2 = 0. \tag{21}$$

Substitution of $(20)_2$ into (21) leads to

$$\mathrm{Im}\,[\mathbf{B}\langle \ln(x_1 - p_* d)\rangle \mathbf{q}_0] + \mathrm{Im}\sum_{\beta=1}^{3}[\ln(x_1 - \bar{p}_\beta d)\mathbf{B}\mathbf{q}_\beta] = 0. \tag{22}$$

The first term can be replaced by the negative of its complex conjugate, i.e.

$$\mathrm{Im}\,[\mathbf{B}\langle \ln(x_1 - p_* d)\rangle \mathbf{q}_0] = -\mathrm{Im}\,[\bar{\mathbf{B}}\langle \ln(x_1 - \bar{p}_* d)\rangle \bar{\mathbf{q}}_0],$$

in which

$$\langle \ln(x_1 - \bar{p}_* d)\rangle = \sum_{\beta=1}^{3}\ln(x_1 - \bar{p}_\beta d)\mathbf{I}_\beta,$$

and

$$\mathbf{I}_1 = \mathrm{diag}\,[1,0,0], \quad \mathbf{I}_2 = \mathrm{diag}\,[0,1,0], \quad \mathbf{I}_3 = \mathrm{diag}\,[0,0,1]. \tag{23}$$

A trivial identity is

$$\sum_{\beta=1}^{3} \mathbf{I}_\beta = \mathbf{I}. \tag{24}$$

Equation (22) now reduces to

$$\mathbf{B}\mathbf{q}_\beta = \bar{\mathbf{B}}\mathbf{I}_\beta\bar{\mathbf{q}}_0, \tag{25}$$

which gives

$$\mathbf{q}_\beta = \mathbf{B}^{-1}\bar{\mathbf{B}}\mathbf{I}_\beta\bar{\mathbf{q}}_0. \tag{26}$$

If the boundary $x_2 = 0$ is a rigid surface, then

$$\mathbf{u} = 0, \quad \text{at}\, x_2 = 0.$$

The same procedure shows that the solution is given by (20) with

$$\mathbf{q}_\beta = \mathbf{A}^{-1}\bar{\mathbf{A}}\mathbf{I}_\beta\bar{\mathbf{q}}_0. \tag{27}$$

Equations (26) and (27) have also been obtained by Suo (1990).

4. IMAGE FORCE IN THE HALF-SPACE

The image force F on the line dislocation induced by the boundary $x_2 = 0$ is

$$F = -\sigma_{i1}^d b_i = \mathbf{b}^T \boldsymbol{\phi}_{,2}^d, \tag{28}$$

where F is in the direction of the x_2-axis if it is positive and σ_{i1}^d is the stress deduced from the second terms in $(20)_2$ evaluated at $x_1 = 0$, $x_2 = d$. We have, using (23),

$$\begin{aligned} F &= \frac{1}{\pi d}\operatorname{Im}\sum_{\beta=1}^{3} \mathbf{b}^T\left[\mathbf{B}\left\langle \frac{p_*}{p_* - \bar{p}_\beta}\right\rangle \mathbf{q}_\beta\right] \\ &= \frac{1}{\pi d}\operatorname{Im}\sum_{\beta=1}^{3}\sum_{\alpha=1}^{3} \mathbf{b}^T\left[\frac{p_\alpha}{p_\alpha - \bar{p}_\beta}\mathbf{B}\mathbf{I}_\beta\mathbf{q}_\beta\right]. \end{aligned} \tag{29}$$

When the boundary is traction-free, substitution of (26) with $\mathbf{q}_0 = \mathbf{B}^T\mathbf{b}$ into (29) leads to

$$F = \frac{-1}{2\pi d}\operatorname{Re}\sum_{\beta=1}^{3}\sum_{\alpha=1}^{3} \mathbf{b}^T\left[\frac{p_\alpha}{p_\alpha - \bar{p}_\beta}(\mathbf{B}\mathbf{I}_\alpha\mathbf{B}^{-1})\mathbf{L}(\bar{\mathbf{B}}\mathbf{I}_\beta\bar{\mathbf{B}}^{-1})^T\right]\mathbf{b}, \tag{30}$$

where use has been made of $(13)_3$. We now perform the following three operations on the right of (30) which leave F unchanged. First, the α and β are interchanged. Next, the right-hand side is replaced by its complex conjugate. Finally, we apply the matrix transpose on the right. The result is

$$F = \frac{-1}{2\pi d}\operatorname{Re}\sum_{\beta=1}^{3}\sum_{\alpha=1}^{3} \mathbf{b}^T\left[\frac{-\bar{p}_\beta}{p_\alpha - \bar{p}_\beta}(\mathbf{B}\mathbf{I}_\alpha\mathbf{B}^{-1})\mathbf{L}(\bar{\mathbf{B}}\mathbf{I}_\beta\bar{\mathbf{B}}^{-1})^T\right]\mathbf{b}. \tag{31}$$

The addition of (30) and (31) yields

$$F = \frac{-1}{4\pi d} \mathrm{Re} \sum_{\beta=1}^{3} \sum_{\alpha=1}^{3} \boldsymbol{\ell}^{\mathrm{T}} [(\mathbf{B}\mathbf{I}_\alpha \mathbf{B}^{-1})\mathbf{L}(\bar{\mathbf{B}}\mathbf{I}_\beta \bar{\mathbf{B}}^{-1})^{\mathrm{T}}]\boldsymbol{\ell},$$

or, by (24),

$$F = \frac{-1}{4\pi d} \boldsymbol{\ell}^{\mathrm{T}} \mathbf{L} \boldsymbol{\ell}. \tag{32}$$

Since $\mathbf{L}$ is positive definite, $F < 0$. This means that the image force always attracts the dislocation to the free boundary.

If the boundary $x_2 = 0$ is a fixed boundary, insertion of (27) in (29) with $\mathbf{q}_0 = \mathbf{B}^{\mathrm{T}}\boldsymbol{\ell}$ and making use of $(13)_2$ gives

$$F = \frac{1}{2\pi d} \mathrm{Re} \sum_{\beta=1}^{3} \sum_{\alpha=1}^{3} \boldsymbol{\ell}^{\mathrm{T}} \left[\frac{p_\alpha}{p_\alpha - \bar{p}_\beta} (\mathbf{B}\mathbf{I}_\alpha \mathbf{A}^{-1})\mathbf{H}(\bar{\mathbf{B}}\mathbf{I}_\beta \bar{\mathbf{A}}^{-1})^{\mathrm{T}} \right] \boldsymbol{\ell}. \tag{33}$$

Following the derivation of (31) from (30), (33) can be rewritten as

$$F = \frac{1}{2\pi d} \mathrm{Re} \sum_{\beta=1}^{3} \sum_{\alpha=1}^{3} \boldsymbol{\ell}^{\mathrm{T}} \left[\frac{-\bar{p}_\beta}{p_\alpha - \bar{p}_\beta} (\mathbf{B}\mathbf{I}_\alpha \mathbf{A}^{-1})\mathbf{H}(\bar{\mathbf{B}}\mathbf{I}_\beta \bar{\mathbf{A}}^{-1})^{\mathrm{T}} \right] \boldsymbol{\ell}. \tag{34}$$

Addition of (33) and (34) leads to

$$F = \frac{1}{4\pi d} \boldsymbol{\ell}^{\mathrm{T}} \mathrm{Re}\, [(\mathbf{B}\mathbf{A}^{-1})\mathbf{H}(\bar{\mathbf{B}}\bar{\mathbf{A}}^{-1})^{\mathrm{T}}]\boldsymbol{\ell}$$

or, using $(15a)_2$,

$$F = \frac{1}{4\pi d} \boldsymbol{\ell}^{\mathrm{T}} [\mathbf{H}^{-1} - \mathbf{H}^{-1}\mathbf{S}\mathbf{S}]\boldsymbol{\ell}. \tag{35}$$

By writing (14) as

$$\mathbf{L} - \mathbf{H}^{-1}\mathbf{S}\mathbf{S} = \mathbf{H}^{-1} \tag{36}$$

we have the alternative form

$$F = \frac{1}{4\pi d} \boldsymbol{\ell}^{\mathrm{T}} [2\mathbf{H}^{-1} - \mathbf{L}]\boldsymbol{\ell}. \tag{37}$$

We will show that $F > 0$, i.e. the image force tends to push the dislocation away from the boundary $x_2 = 0$. To this end, we rearrange (36) as, noting that $\mathbf{H}^{-1}\mathbf{S}$ is antisymmetric,

$$\mathbf{H}^{-1} - \mathbf{L} = \mathbf{S}^{\mathrm{T}}\mathbf{H}^{-1}\mathbf{S}.$$

Since $\mathbf{H}^{-1}$ is positive definite while $\mathbf{S}$ is singular (Barnett and Lothe, 1973; Chadwick and Ting, 1987), $\mathbf{S}^{\mathrm{T}}\mathbf{H}^{-1}\mathbf{S}$ is positive semi-definite. Therefore $(\mathbf{H}^{-1} - \mathbf{L})$ is positive semi-definite and, when (37) is written as

$$F = \frac{1}{4\pi d} \{2\boldsymbol{\ell}^{\mathrm{T}}(\mathbf{H}^{-1} - \mathbf{L})\boldsymbol{\ell} + \boldsymbol{\ell}^{\mathrm{T}}\mathbf{L}\boldsymbol{\ell}\},$$

we have

$$F \geqslant \frac{1}{4\pi d} \boldsymbol{b}^{\mathrm{T}} \mathbf{L} \boldsymbol{b}. \tag{38}$$

From (32), the right-hand side of (38) is the magnitude of the image force when the surface $x_2 = 0$ is traction-free. Thus the image force for a fixed boundary is equal to or larger than the image force for a traction-free boundary. The equality in (38) holds only when $\boldsymbol{b}$ is the right null vector of $\mathbf{S}$, i.e. when

$$\mathbf{S}\boldsymbol{b} = \mathbf{0}.$$

For monoclinic materials with the symmetry plane at $x_3 = 0$, the right null vector is in the direction of the x_3-axis (Ting, 1992a).

Under orthogonal transformations, the Burgers vector $\boldsymbol{b}$ transforms as a tensor of rank one. The tensors $\mathbf{S}$, $\mathbf{H}$ and $\mathbf{L}$ are tensors of rank two when the transformations are limited to rotations about the x_3-axis (Ting, 1982, 1991). Therefore, the image forces F given in (32) and (37) are invariant under rotations about the x_3-axis. The image force is also invariant, obviously, with translation of the coordinate system. Thus one reaches the conclusion that the image force for half-spaces remains the same if the boundary $x_2 = 0$ is replaced by any plane boundary which is parallel to the x_3-axis such that the distance between the line dislocation and the new boundary has the same value d. For isotropic materials and for transversely isotropic materials with the axis of symmetry along the x_3-axis, this means that, if the boundary remains at $x_2 = 0$, the image force is unchanged when we rotate the Burgers vector $\boldsymbol{b}$ about the x_3-axis.

5. IMAGE FORCE FOR BIMATERIALS

Let the half-space $x_2 > 0$ be occupied by material 1 and the other half-space $x_2 < 0$ be occupied by material 2. They are perfectly bonded together so that

$$\mathbf{u}_1 = \mathbf{u}_2, \quad \boldsymbol{\phi}_1 = \boldsymbol{\phi}_2 \quad \text{at } x_2 = 0. \tag{39}$$

The subscripts 1 and 2 for $\mathbf{u}$ and $\boldsymbol{\phi}$ refer to materials 1 and 2, respectively. For a line force $\boldsymbol{f}$ and line dislocation with Burgers vector $\boldsymbol{b}$ applied at the location given by (19), the solution can be written as

$$\begin{aligned}
\mathbf{u}_1 &= \frac{1}{\pi}\operatorname{Im}[\mathbf{A}_1\langle \ln(z_*^{(1)} - p_*^{(1)}d)\rangle \mathbf{q}_0] + \frac{1}{\pi}\operatorname{Im}\sum_{\beta=1}^{3}[\mathbf{A}_1\langle \ln(z_*^{(1)} - \bar{p}_\beta^{(1)}d)\rangle \mathbf{q}_\beta^{(1)}], \\
\boldsymbol{\phi}_1 &= \frac{1}{\pi}\operatorname{Im}[\mathbf{B}_1\langle \ln(z_*^{(1)} - p_*^{(1)}d)\rangle \mathbf{q}_0] + \frac{1}{\pi}\operatorname{Im}\sum_{\beta=1}^{3}[\mathbf{B}_1\langle \ln(z_*^{(1)} - \bar{p}_\beta^{(1)}d)\rangle \mathbf{q}_\beta^{(1)}],
\end{aligned} \tag{40}$$

for material 1 in $x_2 > 0$ and

$$\begin{aligned}
\mathbf{u}_2 &= \sum_{\beta=1}^{3}\frac{1}{\pi}\operatorname{Im}[\mathbf{A}_2\langle \ln(z_*^{(2)} - p_\beta^{(1)}d)\rangle \mathbf{q}_\beta^{(2)}], \\
\boldsymbol{\phi}_2 &= \sum_{\beta=1}^{3}\frac{1}{\pi}\operatorname{Im}[\mathbf{B}_2\langle \ln(z_*^{(2)} - p_\beta^{(1)}d)\rangle \mathbf{q}_\beta^{(2)}],
\end{aligned} \tag{41}$$

for material 2 in $x_2 < 0$. The subscripts 1, 2 or the superscripts (1), (2) denote the quantity in material 1 and material 2, respectively. $\mathbf{q}_0$ is given in (18) and $\mathbf{q}_\beta^{(1)}, \mathbf{q}_\beta^{(2)}$ are unknown constant vectors which are determined by substituting (40), (41) into (39). Following the derivation of (25) we obtain

$$\mathbf{A}_1\mathbf{q}_\beta^{(1)}+\bar{\mathbf{A}}_2\bar{\mathbf{q}}_\beta^{(2)} = \bar{\mathbf{A}}_1\mathbf{I}_\beta\bar{\mathbf{q}}_0,$$
$$\mathbf{B}_1\mathbf{q}_\beta^{(1)}+\bar{\mathbf{B}}_2\bar{\mathbf{q}}_\beta^{(2)} = \bar{\mathbf{B}}_1\mathbf{I}_\beta\bar{\mathbf{q}}_0. \tag{42}$$

Rewriting $(42)_1$ as

$$(\mathbf{A}_1\mathbf{B}_1^{-1})(\mathbf{B}_1\mathbf{q}_\beta^{(1)})+(\bar{\mathbf{A}}_2\bar{\mathbf{B}}_2^{-1})(\bar{\mathbf{B}}_2\bar{\mathbf{q}}_\beta^{(2)}) = (\bar{\mathbf{A}}_1\bar{\mathbf{B}}_1^{-1})(\bar{\mathbf{B}}_1\mathbf{I}_\beta\bar{\mathbf{q}}_0)$$

and eliminating $\mathbf{B}_1\mathbf{q}_\beta^{(1)}$ from $(42)_2$ one obtains

$$(\mathbf{A}_1\mathbf{B}_1^{-1}-\bar{\mathbf{A}}_2\bar{\mathbf{B}}_2^{-1})(\bar{\mathbf{B}}_2\bar{\mathbf{q}}_\beta^{(2)}) = (\mathbf{A}_1\mathbf{B}_1^{-1}-\bar{\mathbf{A}}_1\bar{\mathbf{B}}_1^{-1})(\bar{\mathbf{B}}_1\mathbf{I}_\beta\bar{\mathbf{q}}_0),$$

or by (15b),

$$(\mathbf{M}_1^{-1}+\bar{\mathbf{M}}_2^{-1})(\bar{\mathbf{B}}_2\bar{\mathbf{q}}_\beta^{(2)}) = 2\mathbf{L}_1^{-1}(\bar{\mathbf{B}}_1\mathbf{I}_\beta\bar{\mathbf{q}}_0).$$

Therefore

$$\bar{\mathbf{B}}_2\bar{\mathbf{q}}_\beta^{(2)} = 2(\mathbf{M}_1^{-1}+\bar{\mathbf{M}}_2^{-1})^{-1}\mathbf{L}_1^{-1}(\bar{\mathbf{B}}_1\mathbf{I}_\beta\bar{\mathbf{q}}_0), \tag{43a}$$

and from $(42)_2$,

$$\mathbf{B}_1\mathbf{q}_\beta^{(1)} = [\mathbf{I}-2(\mathbf{M}_1^{-1}+\bar{\mathbf{M}}_2^{-1})^{-1}\mathbf{L}_1^{-1}](\bar{\mathbf{B}}_1\mathbf{I}_\beta\bar{\mathbf{q}}_0). \tag{43b}$$

This completes the derivation of the Green's function for bimaterials (Ting, 1992b).

With (15b) we may write

$$\mathbf{M}_1^{-1}+\bar{\mathbf{M}}_2^{-1} = \mathbf{D}-i\mathbf{W}, \tag{44}$$

where

$$\mathbf{D} = \mathbf{L}_1^{-1}+\mathbf{L}_2^{-1}, \quad \mathbf{W} = \mathbf{S}_1\mathbf{L}_1^{-1}-\mathbf{S}_2\mathbf{L}_2^{-1}. \tag{45}$$

Let

$$(\mathbf{M}_1^{-1}+\bar{\mathbf{M}}_2^{-1})^{-1} = (\tilde{\mathbf{D}}+i\tilde{\mathbf{W}}), \tag{46}$$

where $\tilde{\mathbf{D}}$ and $\tilde{\mathbf{W}}$ are real. We then have

$$(\tilde{\mathbf{D}}+i\tilde{\mathbf{W}})(\mathbf{D}-i\mathbf{W}) = \mathbf{I} = (\mathbf{D}-i\mathbf{W})(\tilde{\mathbf{D}}+i\tilde{\mathbf{W}}).$$

Equating the real and imaginary parts of the equations it can be shown that

$$\tilde{\mathbf{D}} = (\mathbf{D}+\mathbf{W}\mathbf{D}^{-1}\mathbf{W})^{-1}, \quad \tilde{\mathbf{W}} = \mathbf{D}^{-1}\mathbf{W}\tilde{\mathbf{D}} = \tilde{\mathbf{D}}\mathbf{W}\mathbf{D}^{-1}.$$

Hence (43b) has the expression

$$\mathbf{B}_1\mathbf{q}_\beta^{(1)} = [\mathbf{I}-2(\tilde{\mathbf{D}}+i\tilde{\mathbf{W}})\mathbf{L}_1^{-1}](\bar{\mathbf{B}}_1\mathbf{I}_\beta\bar{\mathbf{q}}_0). \tag{47}$$

The image force is obtained by substituting the second terms of $\boldsymbol{\phi}_1$ in (40) into $(22)_2$ and evaluating the result at $x_1 = 0$, $x_2 = \mathrm{d}$. After using (47) with $\mathbf{q}_0 = \mathbf{B}^{\mathrm{T}}\mathscr{E}$ and following the derivation of (26), we obtain

$$F = \frac{-1}{4\pi d}\boldsymbol{b}^{\mathrm{T}}[\mathbf{L}_1 - 2\tilde{\mathbf{D}}]\boldsymbol{b}. \tag{48}$$

Again, F is invariant with rotations about the x_3-axis. Thus the image force remains the same if the interface $x_2 = 0$ is replaced by any plane which is parallel to the x_3-axis and the distance between the line dislocation and the new interface remains the same, namely d (Barnett and Lothe, 1974; Tucker, 1969).

We will now deduce (26) and (31) from (48). Consider the special bimaterial for which

$$C^{(2)}_{ijks} = \lambda C^{(1)}_{ijks}, \quad \lambda \geqslant 0, \tag{49}$$

where λ is an arbitrary real positive constant. $\lambda = 0$ when material 2 is a vacuum, $\lambda = 1$ if the two materials are identical, and $\lambda = \infty$ when material 2 is a rigid body. From (4) the un-normalized eigenvectors $\mathbf{a}$ for materials 1 and 2 can be taken to be identical and, by $(10)_1$, $\mathbf{A}_2 = \mathbf{A}_1$. When normalized, however, $\mathbf{A}_2$ should be taken as proportional to $\mathbf{A}_1$, i.e.

$$\mathbf{A}_2 = k\mathbf{A}_1,$$

where k is the proportionality factor. From (7) and $(10)_2$,

$$\mathbf{B}_2 = \lambda k \mathbf{B}_1$$

and the orthogonality relation (12) yields

$$k = \lambda^{-1/2}.$$

We therefore have the result that, when (49) holds,

$$\mathbf{A}_2 = \lambda^{-1/2}\mathbf{A}_1, \quad \mathbf{B}_2 = \lambda^{1/2}\mathbf{B}_1. \tag{50}$$

From $(15\mathrm{b})_1$,

$$\mathbf{M}_2^{-1} = \frac{1}{\lambda}\mathbf{M}_1^{-1}$$

and (46) becomes

$$(\tilde{\mathbf{D}} + \mathrm{i}\tilde{\mathbf{W}}) = \left(\mathbf{M}_1^{-1} + \frac{1}{\lambda}\bar{\mathbf{M}}^{-1}\right)^{-1} = \lambda(\lambda\mathbf{M}_1^{-1} + \bar{\mathbf{M}}_1^{-1})^{-1}. \tag{51}$$

When the boundary $x_2 = 0$ is traction-free, material 2 is a vacuum, and setting $\lambda = 0$ in $(51)_2$ leads to

$$\tilde{\mathbf{D}} + \mathrm{i}\tilde{\mathbf{W}} = \mathbf{0}$$

so that (48) reduces to (32). If the boundary $x_2 = 0$ is a fixed boundary, material 2 can be considered as infinitely rigid, whereupon setting $\lambda = \infty$ in $(51)_1$ yields

$$\tilde{\mathbf{D}} + \mathrm{i}\tilde{\mathbf{W}} = \mathbf{M}_1 = \mathbf{H}_1^{-1} + \mathrm{i}\mathbf{H}_1^{-1}\mathbf{S}_1$$

by $(15\mathrm{a})_2$. Equation (48) reduces to (37).

In closing this section we point out that, when (49) holds, substitution of (50) into (13) leads to

$$\mathbf{S}_2 = \mathbf{S}_1, \quad \mathbf{H}_2 = \frac{1}{\lambda}\mathbf{H}_1, \quad \mathbf{L}_2 = \lambda \mathbf{L}_1, \tag{52}$$

and (45) yields the relations

$$\mathbf{D} = \frac{\lambda+1}{\lambda}\mathbf{L}_1^{-1}, \quad \mathbf{W} = \frac{\lambda-1}{\lambda}\mathbf{S}_1\mathbf{L}_1^{-1}. \tag{53}$$

Hence, noticing that $\mathbf{S}_1\mathbf{L}_1^{-1}$ is antisymmetric,

$$\mathbf{D}^{-1}\mathbf{W} = \frac{1-\lambda}{1+\lambda}\mathbf{S}_1^{\mathrm{T}} \tag{54}$$

which is bounded for all positive λ. Equations (50)–(54) may be useful for specializing other bimaterial problems to problems involving only one material whose boundary can be either traction-free or rigidly clamped. It is interesting to note from $(52)_1$ that $\mathbf{S}_2 = \mathbf{S}_1$ for any value of λ when (49) holds.

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